MTH241 Fall 2024: Exam 02

Instructor: Dr. Ling Liang

Date: October 23, 2024

Time: 3:00pm - 3:50pm

Name:
UID:

Closed book, no calculator, show your work clearly.

1. (15pt) Mark True or False for each of the following statements. (**Grading:** 3pt each)

   (a) Let $f(x, y)$ be a function in two variables. Then the graph of f is the same as the level surface of the function $g(x, y, z) = f(x, y) - z$ with $c = 0$.

   Your answer: True False

   (b) Let R be a set in the plane, then R must have an interior point.

   Your answer: True False

   (c) Let $f(x, y)$ be a function in two variables, then $f_{xy}(x, y) = f_{yx}(x, y)$ always holds.

   Your answer: True False

   (d) Let f have a relative extreme value at (x_0, y_0), then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

   Your answer: True False

   (e) Let f be a continuous function on a bounded set R, then f has both a maximum value and minimum value on R.

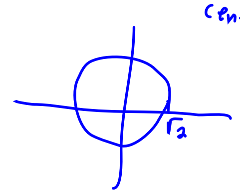
   Your answer: True False

2. (35pt) Let $f(x, y) = \sqrt{5 - 2x^2 - 2y^2}$.

   (a) Compute the domain of f and express the level curve of f with $c = 1$ using set notation. What is the shape of this level curve? (**Grading:** 3pt for working, 3pt for correct domain, 3pt for correct level curve, 1pt for correct shape.)

Domain $= \{(x,y): 5-2x^2-2y^2 \geq 0\} = \{(x,y): x^2+y^2 \leq \frac{5}{2}\}$

Level curve $= \{(x,y): \sqrt{5-2x^2-2y^2} = 1\} = \{(x,y): 5-2x^2-2y^2 = 1\} = \{(x,y): x^2+y^2 = 2\}$ = circle of radius $\sqrt{2}$ center $(0,0)$



(b) Let $\vec{a} = 3\vec{i} - 4\vec{j}$, compute $D_{\vec{a}} f(1,1)$. In what direction is f increasing most rapidly at $(1,1)$? (**Grading:** 4pt for working, 3pt for each correct answer.)

$$f = \sqrt{5-2x^2-2y^2}$$

$$\nabla f = \begin{bmatrix} \frac{-2x}{\sqrt{5-2x^2-2y^2}} \\ \frac{-2y}{\sqrt{5-2x^2-2y^2}} \end{bmatrix}$$

$$\nabla f(1,1) = \begin{bmatrix} \frac{-2}{1} \\ \frac{-2}{1} \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$D_a f = \nabla f \cdot \frac{a}{\|a\|} (1,1) = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \frac{1}{\sqrt{3^2+4^2}}$$

$$= (-6+8) \cdot \frac{1}{5} = \frac{2}{5}$$

f is increasing most rapidly at $u = \nabla f(1,1)$

$$= \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

(c) Find a normal vector to the graph of f at $(-1,-1,1)$, and find an equation of the plane tangent to the graph of f at $(-1,-1,1)$. (**Grading:** 5pt for working, 5pt for correct normal vector, 5pt for correct tangle plane)

Let $g(x,y,z) = z - f(x,y) = z - \sqrt{5-2x^2-2y^2}$

Then $N = \nabla g(-1,-1,1) = \begin{bmatrix} \frac{2x}{\sqrt{5-2x^2-2y^2}} \\ \frac{2y}{\sqrt{5-2x^2-2y^2}} \\ 1 \end{bmatrix}_{(-1,-1,1)} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ = normal vector

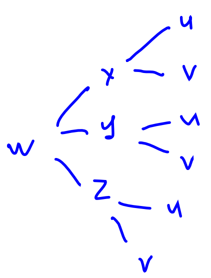
$P = (-1,-1,1)$

$E:\ 2(x+1) + 2(y+1) + (z-1) = 0$ tangent plane

3. (15pt) Consider the following function:

$$w = f(x, y, z) = x + y + z, \quad x = e^{uv}, \quad y = \sin(u + v), \quad z = u^2 - v^2.$$

Compute the partial derivative $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ using the chain rule. (**Grading:** 5pt for working, 5pt for each partial derivative)



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}.$$

$$= 1 \cdot ve^{uv} + 1 \cdot \cos(u+v) + 1 \cdot 2u$$

$$\frac{\partial w}{\partial v} = 1 \cdot ue^{uv} + 1 \cdot \cos(u+v) - 1 \cdot 2v$$

4. (10pt) Approximate $\sqrt{2.98^2 + 4.04^2}$ using the tangent plane approximation. Simplify your final answer. (**Grading:** 5pt for working, 5pt for correct answer)

$f(x,y) = \sqrt{x^2+y^2}$

$P = (2.98, 4.04)$

$Q = (3, 4)$

$g = z - \sqrt{x^2+y^2}$

$$N = \nabla g\,(Q) = \begin{bmatrix} \frac{-x}{\sqrt{x^2+y^2}} \\ \frac{-y}{\sqrt{x^2+y^2}} \\ 1 \end{bmatrix}_{(3,4)} = \begin{bmatrix} \frac{-3}{5} \\ \frac{-4}{5} \\ 1 \end{bmatrix}$$

For simplicity take $N' = 5N = (-3, -4, 5)$ } $\Rightarrow E: -3(x-3) - 4(y-4) + 5(z-5) = 0$

$Q = (3, 4) \rightsquigarrow z = f(3,4) = 5 \Rightarrow Q = (3, 4, 5)$

Evaluated at $P = (2.98, 4.04)$; $-3(2.98-3) - 4(4.04-4) + 5(z-5) = 0$

$\Rightarrow +3 \cdot 0.02 - 4 \cdot 0.04 + 5(z-5) = 0 \Rightarrow z = \frac{-3 \cdot 0.02 + 4 \cdot 0.04}{5} + 5$

$= \frac{0.1}{5} + 5 \boxed{= 5.02}$

5. (25pt) Let $f(x, y) = 2x^2 - 4x + 3y^2 + 1$.

(a) Use the Second Partials Test to determine at which points f has relative extreme values (if any) and at which points f has saddle points (if any). (**Grading:** 5pt for working, 5pt for second partials test)

$$\nabla f = \begin{bmatrix} 4x-4 \\ 6y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad x=1 \text{ and } y=0 \Rightarrow P=(1,0) \text{ crit pt}$$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \rightsquigarrow \left.\begin{matrix} \det(H) = 24 > 0 \\ f_{xx} = 4 > 0 \end{matrix}\right\} \Rightarrow P = \text{local min}$$

(b) Find all the extreme values of f on the disk $x^2 + y^2 \leq 16$ by the Lagrange method. (**Grading:** 5pt for working, 10pt for extreme values)

- $P=(1,0)$ crit pt on interior from (a)
- Boundary: Use Lagrange: $f(x,y) = 2x^2 - 4x + 3y^2 + 1$, $g(x,y) = x^2 + y^2$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{bmatrix} 4x-4 \\ 6y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Lagrange: $\begin{cases} 4x-4 = 2\lambda x \\ 6y = 2\lambda y \\ x^2+y^2 = 16 \end{cases} \rightsquigarrow y(6-2\lambda) = 0 \Rightarrow y=0 \text{ or } \lambda = 3$

Case 1: $\boxed{y=0}$ Then $x^2 + 0^2 = 16 \Rightarrow x = \pm 4 \Rightarrow \boxed{P_2 = (4,0)}, \boxed{P_3 = (-4,0)}$

Case 2: $\lambda = 3$: Then $4x - 4 = 6x \Rightarrow \boxed{x=-2} \Rightarrow (-2)^2 + y^2 = 16 \Rightarrow y = \pm\sqrt{12} \Rightarrow \boxed{P_4 = (-2, \sqrt{12})}$, $\boxed{P_5 = (-2, -\sqrt{12})}$

Evaluate and compare: $f(P_1) = 2 - 4 + 1 = -1$ min

$f(P_2) = 2 \cdot 16 - 16 + 1 = 17$

$f(P_3) = 2 \cdot 16 + 16 + 1 = 49$

$f(P_4) = 2 \cdot 4 + 4 \cdot 2 + 3 \cdot 12 + 1 = 53$ max

$f(P_5) = f(P_4) = 53$

Extra page: